Solutions for Week 1

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1 Exercise 1.1

We wish to demonstrate the bounds on the Binomial coefficient used in the derivation at the beginning of the chapter:

$$\frac{e^{nH(k/m)}}{n+1} < \binom{n}{k} = \frac{n!}{k!(n-k)!} < e^{nH(k/n)} \tag{1.1}$$

where $H(p) = -p\log p - (1-p)\log(1-p)$ is the Binomial entropy (we used $p = (1+m)/2$ in the chapter).

(a) Use the Binomial to prove that, for any $0 < p < 1$:

$$1 = \sum_{i=0}^{n} \binom{n}{i} (1-p)^i p^{n-i} \tag{1.2}$$

Solution: Recall the *Binomial theorem* (see Appendix A for a proof):

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \tag{1.3}$$

valid for any $x, y \in \mathbb{R}$ and $n \geq 0$. Letting $x = 1-p$ and $y = p$ leads to the desired result:

$$(1-p+p)^n = 1^n = 1 \overset{!}{=} \sum_{i=0}^{n} \binom{n}{i} (1-p)^i p^{n-i}. \tag{1.4}$$

(b) Using a particular value of p, and keeping only one term of the sum, show that

$$\binom{n}{k} < e^{nH(k/n)}$$

Solution: First note that we can rewrite:

$$\begin{aligned} H(p) &= -p\log p - (1-p)\log(1-p) = -p\left(\log p - \log(1-p)\right) - \log(1-p) \\ &= -p\log\left(\frac{p}{1-p}\right) - \log(1-p) \end{aligned} \tag{1.5}$$

In particular, for $p = k/n$:

$$e^{nH(k/n)} = \left(1 - \frac{k}{n}\right)^{-n+k} \left(\frac{k}{n}\right)^{-k} \qquad \Leftrightarrow \qquad \left(1 - \frac{k}{n}\right)^{n-k} \left(\frac{k}{n}\right)^{k} e^{nH(k/n)} = 1 \tag{1.6}$$

Therefore, using the result from point (a) also with $p = k/n$ $(0 \leq k \leq n)$, we can write:

$$\left(1 - \frac{k}{p}\right)^{n-k} \left(\frac{k}{n}\right)^{k} e^{nH(k/n)} = \sum_{i=0}^{n} \binom{n}{i} \left(1 - \frac{k}{n}\right)^{i} \left(\frac{k}{n}\right)^{n-i} \tag{1.7}$$

To get to our result, we just need to notice that in a sum of non-negative terms, the sum is always larger than any of the summands. Or in mathematical terms: for any non-negative function $f : \{0, \cdots n\} \to \mathbb{R}$ we have:

$$\sum_{i=0}^{n} f(i) \geq f(k), \qquad \text{for any } 0 \leq k \leq n. \tag{1.8}$$

Applying this to our sum for the term $i = k$:

$$\left(1 - \frac{k}{p}\right)^{n-k} \left(\frac{k}{n}\right)^{k} e^{nH(k/n)} \geq \binom{n}{k} \left(1 - \frac{k}{n}\right)^{k} \left(\frac{k}{n}\right)^{n-k} \tag{1.9}$$

which simplifies to the desired result:

$$e^{nH(k/n)} \geq \binom{n}{k} \tag{1.10}$$

(c) If one makes n draws from a binomial variable with probability of positive outcome $p = k/n$, what is the most probable value for number of positive? Deduce that

$$(n+1)\binom{n}{k} > e^{nH(k/n)}, \qquad \text{for any } 0 \leq p \leq 1.$$

Solution: Similarly to the above, note the following bound for a sum of non-negative terms $f : \{0, \cdots, n\} \to \mathbb{R}$:

$$\sum_{i=0}^{n} f(i) \leq (n+1) \max_{0 \leq i \leq n} f(i) \tag{1.11}$$

In our case, we have:

$$f(i) = \binom{n}{i} p^{i} (1-p)^{n-i} \tag{1.12}$$

which is the probability mass function (pmf) of a binomial random variable with probability p. It is a known fact from probability that the mode (most probable value) of a binomial random variable $X \sim \mathrm{Binom(n,p)}$ is equal to its expected value $\mathbb{E}X = np$. You can convince yourself of this by plotting $f(i)$, or you can check the proof of this fact in Appendix B. In any case, applying this to our sum gives:

$$1 = \sum_{i=1}^{n} \binom{n}{i} (1-p)^{i} p^{n-i} \leq (n+1) \max_{0 \leq i \leq n} \left[\binom{n}{i} (1-p)^{i} p^{n-i}\right] = (n+1)\binom{n}{\lfloor np \rfloor} (1-p)^{\lfloor np \rfloor} p^{n - \lfloor np \rfloor} \tag{1.13}$$

In particular, for $p = k/n$ with $0 \leq k \leq n$:

$$1 \leq (n+1)\binom{n}{k} \left(1 - \frac{k}{n}\right)^{k} \left(\frac{k}{n}\right)^{n-k} \tag{1.14}$$

Using again the result in eq. (1.6) and simplifying both sides yield the result:

$$\frac{1}{n+1} e^{nH(k/n)} \leq \binom{n}{k} \tag{1.15}$$

2 Exercise 1.2

In this exercise, we will be interested in the asymptotic behaviour ($\lambda \to \infty$) of the following class of real integrals:

$$I(\lambda) = \int_a^b \mathrm{d}t\ h(t)e^{\lambda f(t)} \tag{2.1}$$

(a) Intuitively, what are the regions in the interval $[a, b]$ which will contribute more to the value of $I(\lambda)$?

Solution: The integral is dominated by the regions in $[a, b]$ in which the integrand $h(t)e^{\lambda f(t)}$ is bigger. For $\lambda \gg 1$, the exponential term dominates, assigning more mass to the regions in which the function $f(t)$ is bigger. Therefore, at $\lambda \to \infty$ we expect the integral to be dominated, at leading order, by:

$$c = \underset{t\in[a,b]}{\operatorname{argmax}} f(t) \tag{2.2}$$

See Fig. 1 (left) for a visual example. Note that there are two caveats of this intuition. First, if $h(c) = 0$, the integrand will be exactly zero at c, and the integral will be dominated by the second highest maxima (if any) see Fig. 1 (right). Second, if there are two or more global maxima, one should take into account the contribution of each of them into Laplace's formula.

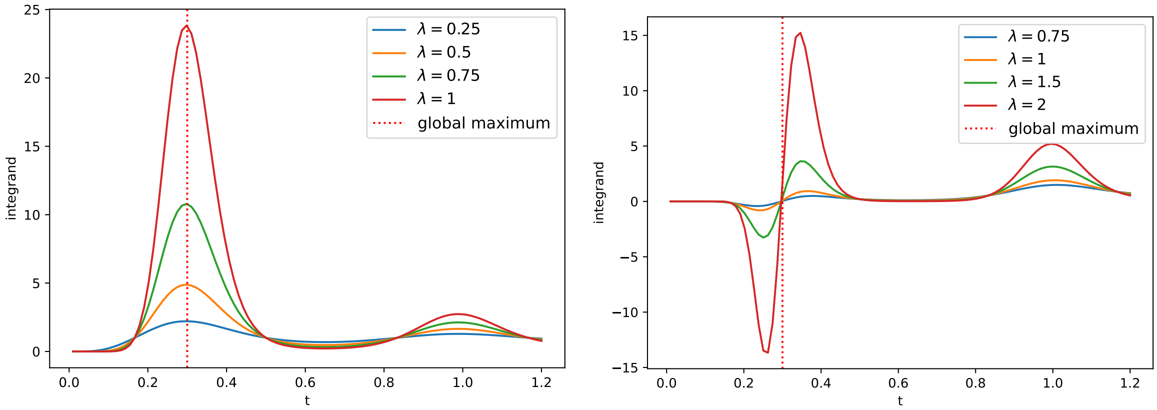


Figure 1: (Left) Integrand of $I(\lambda)$ for $h(t) = 1$ and $f(t) = \cos(3\pi t)/t$, for different values of λ. Note that this function has two maxima, and as $\lambda \to \infty$ the biggest contribution peaks around the global maximum. (Right) Same $f(t)$ as before, but now $h(t) = (t - c)$ such that the integrand is zero at the maximum of $f(t)$. In this case, the integral is dominated by the local maximum at $t = 1$, since the peaks near $t = c$ are symmetric and will give cancelling contributions.

(b) Suppose the function $f(t)$ has a single global maximum at a point $c \in [a, b]$ such that $f''(c) < 0$, and assume $h(c) \neq 0$. Using a Taylor expansion for f, show that for $\lambda \gg 1$ we expect the integral to behave:

$$I(\lambda) \underset{\lambda\gg1}{=} \int_{c-\epsilon}^{c+\epsilon} \mathrm{d}t\ h(c)e^{\lambda\left[f(c)+\frac{1}{2}f''(c)(t-c)^2\right]} \tag{2.3}$$

where $\epsilon > 0$ is a positive but small real number.

Solution: Consider the Taylor expansion of f around c to second order:

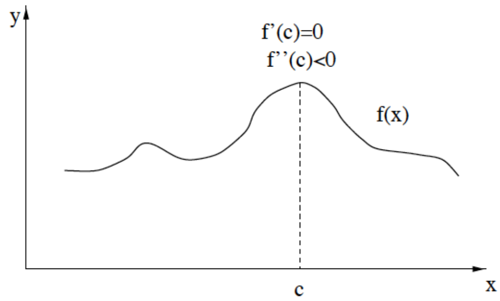
$$f(t) = f(c) + f'(c)(t-c) + \frac{1}{2}f''(c)(t-c)^2 + O\left((t-c)^3\right) \tag{2.4}$$

$$\underset{(a)}{=} f(c) + \frac{1}{2}f''(c)(t-c)^2 + O\left((t-c)^3\right) \tag{2.5}$$

where in (a) we used $f'(c) = 0$. Assuming $h(c) \neq 0$, in a small neighbourhood $(c-\epsilon, c+\epsilon)$ of c with $\epsilon > 0$, from the discussion above we expect the integral to be dominated by the exponential, and therefore we can assume that $h(t) = h(c)$ for $t \in (c-\epsilon, c+\epsilon)$. Applying this to the integral gives the desired result:

$$I(\lambda) \underset{\lambda \gg 1}{=} \int_{c-\epsilon}^{c+\epsilon} \mathrm{d}t\ h(t)e^{\lambda f(t)} \approx \int_{c-\epsilon}^{c+\epsilon} \mathrm{d}t\ h(c)e^{\lambda\left[f(c)+\frac{1}{2}f''(c)(t-c)^2\right]} \tag{2.6}$$

Note that this is a not rigorous derivation. The math inclined student can check a rigorous derivation here.



(c) Using your result above, conclude that:

$$I(\lambda) \asymp \frac{h(c)e^{\lambda f(c)}}{\sqrt{-\lambda f''(c)}} \int_{\mathbb{R}} e^{-t^2}\mathrm{d}t. \tag{2.7}$$

Solution: This is a simple change of variables $x = -\frac{\lambda}{2}f''(c)(t-c)^2$:

$$\begin{aligned} I(\lambda) &\underset{\lambda \gg 1}{=} h(c)e^{\lambda f(c)} \int_{c-\epsilon}^{c+\epsilon} \mathrm{d}t\ e^{\frac{\lambda}{2}f''(c)(t-c)^2} \\ &= h(c)e^{\lambda f(c)} \sqrt{\frac{2}{-\lambda f''(c)}} \int_{-\sqrt{-\lambda f''(c)/2}\epsilon}^{\sqrt{-\lambda f''(c)/2}\epsilon} \mathrm{d}x\ e^{-x^2} \\ &\underset{(a)}{=} h(c)e^{\lambda f(c)} \sqrt{\frac{2}{-\lambda f''(c)}} \int_{\mathbb{R}} \mathrm{d}x\ e^{-x^2} \end{aligned} \tag{2.8}$$

where in (a) we note that as $\lambda \to \infty$ the boundaries go to $\mathbb{R}$ for any fixed finite $\epsilon > 0$.

(d) (Gaussian integral) Show that:

$$\int_{\mathbb{R}} e^{-t^2}\mathrm{d}t = \sqrt{\pi} \tag{2.9}$$

Solution: There are many ways to derive the Gaussian integral. Here we go through a possible solution. The trick consists in computing the square of the left-hand side:

$$I^2 = \left(\int_{\mathbb{R}} \mathrm{d}t\ e^{-t^2}\right)^2 = \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y\ e^{-(x^2+y^2)} \tag{2.10}$$

Now changing to polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$ for $r > 0$ and $\theta \in [0, 2\pi)$ we can rewrite

$$I^2 = \int_0^\infty \mathrm{d}r \int_0^{2\pi} r\mathrm{d}\theta e^{-r^2} = 2\pi \int \mathrm{d}r\ re^{-r^2} = 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty = \pi \tag{2.11}$$

Therefore taking the square root leads to the desired result:

$$I = \int_{\mathbb{R}} \mathrm{d}t\ e^{-t^2} = \sqrt{\pi} \tag{2.12}$$

(e) Deduce Laplace's formula:

$$\int_a^b \mathrm{d}t\ h(t)e^{\lambda f(t)} \asymp \sqrt{\frac{2\pi}{-\lambda f''(c)}} e^{\lambda f(c)} h(c) \tag{2.13}$$

Solution: Doing the Gaussian integral in eq. (2.8) explicitly leads to the desired result:

$$I(\lambda) \underset{\lambda \gg 1}{=} h(c)e^{\lambda f(c)} \sqrt{\frac{2}{-\lambda f''(c)}} \int_{\mathbb{R}} \mathrm{d}x\ e^{-x^2} = h(c)e^{\lambda f(c)} \sqrt{\frac{2\pi}{-\lambda f''(c)}} \tag{2.14}$$

3 Exercise 1.3

The saddle-point method is a generalisation of Laplace's method to the complex plane. As before, we search for an asymptotic formula for integrals of the type:

$$I(\lambda) = \int_\gamma \mathrm{d}z\ h(z)e^{\lambda f(z)} \tag{3.1}$$

where $\gamma : [a,b] \to \mathbb{C}$ is a curve in the complex plane $\mathbb{C}$ and $\lambda > 0$ is a real positive number which we will take to be large. If the complex function f is holomorphic on a connected open set $\Omega \subset \mathbb{C}$, the integral $I(\lambda)$ is independent of the curve γ. The goal is therefore to choose γ wisely.

Part I: Geometrical properties of holomorphic functions

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, and let $z = x + iy \in \mathbb{C}$ for real $x, y \in \mathbb{R}$. Without loss of generality, we can write $f(z) = u(x,y) + iv(v,x)$ for $u, v : \mathbb{R}^2 \to \mathbb{R}$ real-valued functions. The goal of this exercise is to study the properties of f around a critical point $f'(z_0) = 0$ for $z_0 \in \mathbb{C}$.

(a) Show that at a critical point, the gradients of u and v are zero.

Solution: First, note that since f is holomorphic at $z_0 = (x_0, y_0)$, we have:

$$\frac{\mathrm{d}f}{\mathrm{d}\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)(u(x_0,y_0) + iv(x_0,y_0)) \stackrel{!}{=} 0 \tag{3.2}$$

On the other hand, since z_0 is a critical point, we have:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{1}{2}(\partial_x - i\partial_y)(u(x_0,y_0) + iv(x_0,y_0)) \stackrel{!}{=} 0 \tag{3.3}$$

Taking the sum of these two equations lead to:

$$\partial_x u(x_0,y_0) + i\partial_x v(x_0,y_0) = 0 \qquad \Rightarrow \qquad \partial_x u(x_0,y_0) = \partial_x v(x_0,y_0) = 0 \tag{3.4}$$

Similarly, taking the difference between the equations:

$$\partial_y u(x_0,y_0) + i\partial_y v(x_0,y_0) = 0 \qquad \Rightarrow \qquad \partial_y u(x_0,y_0) = \partial_y v(x_0,y_0) = 0 \tag{3.5}$$

(b) Using the Cauchy integral formula, show that for all $z_0 = x_0 + iy_0$ in an open convex set Ω where f is holomorphic we have:

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \; u(x_0 + r\cos\theta, y_0 + r\sin\theta) \tag{3.6}$$

$$v(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \; v(x_0 + r\cos\theta, y_0 + r\sin\theta) \tag{3.7}$$

for all circles of radius $r > 0$ centred at z_0 contained inside Ω. This result is known as the Mean Value Theorem in complex analysis.

Solution: Let $D(r, z_0) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset \Omega$ be a disc with radius $r > 0$ centred at z_0, and denote γ its boundary (i.e. the circle with radius r centered at z_0). Since f is holomorphic in $D(r, z_0)$, for any $a \in D(r, z_0)$ Cauchy integral formula holds:

$$f(a) = \int_\gamma \frac{\mathrm{d}z}{2\pi i}\frac{f(z)}{z-a} \tag{3.8}$$

Note that in particular this is true for $a = z_0$. Parametrising γ as $\gamma(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi)$, we have:

$$\begin{aligned} f(z_0) &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi i} ire^{i\theta}\frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} f(z_0 + re^{i\theta}) \end{aligned} \tag{3.9}$$

Letting $z = (x_0, y_0)$ and $f(x, y) = u(x, y) + iv(x, y)$ and taking the real and imaginary values at each side of the equality yield the expected result.

(c) Conclude that neither u or v can have a local extremum (maximum or minimum) inside Ω. Therefore, all critical points $z_0 \in \Omega$ are necessarily saddle points of u and v.

Solution: The result above tells us that the value of a holomorphic function at the center of a circle equals to its arithmetic mean along the circle. Intuitively, it is easy to see that this implies the function cannot have a maximum or a minimum inside D. Let's show it mathematically.

Suppose that $f'(z_0) = 0$ and that f has a maximum z_0. This implies that for any point on a circle around z_0 with small enough radius $r > 0$ we would have $|f(z_0)| \geq |f(z_0 + re^{i\theta})|$. However, taking the modulus of eq. (3.9):

$$|f(z_0)| = \left| \int_0^{2\pi} \frac{d\theta}{2\pi} f(z_0 + re^{i\theta}) \right| \leq \int_0^{2\pi} \frac{d\theta}{2\pi} \left| f(z_0 + re^{i\theta}) \right| \tag{3.10}$$

which contradicts the fact that z_0 is a maximum unless f is constant. One can show that f cannot have a minimum (unless it is constant) by applying the same argument to $1/f$. Therefore, all critical points of non-constant holomorphic functions are saddles.

(d) Let z_0 be a critical point of f such that $f''(z_0) \neq 0$. Using the Taylor series of f around z_0 and using the polar decomposition $f''(z_0) = \rho e^{i\alpha}$, $z - z_0 = re^{i\theta}$, find the values of $\theta \in [0, 2\pi)$ corresponding to the two directions of steepest-descent of u as a function of α in the complex plane.

Solution: The Taylor expansion of f around the critical point z_0 is given by:

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + O\left((z - z_0)^3\right) \\ &= f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + O\left(|z - z_0|^3\right) \end{aligned} \tag{3.11}$$

Letting $z = z_0 + re^{i\theta}$ and $f''(z_0) = \rho e^{i\alpha}$ for $\theta, \alpha \in [0, 2\pi)$ and $r, \rho > 0$:

$$f(z) - f(z_0) = \frac{1}{2} r^2 \rho e^{i(\alpha + 2\theta)} + O\left(|z - z_0|^3\right) \tag{3.12}$$

Therefore, the rate at which f varies around z_0 is determined by $\frac{1}{2} r^2 \rho$, while the direction is set by the phase $e^{i(\alpha+2\theta)}$. The direction corresponding to the steepest-descent (i.e. faster way down) is given by

$$e^{i(\alpha+2\theta)} = -1, \qquad \Leftrightarrow \qquad \theta = \frac{\pi}{2} - \frac{\alpha}{2} + k\pi, \qquad k \in \mathbb{Z} \tag{3.13}$$

Part II: Choosing the good curve γ

(a) What are the regions of γ which dominate the integral $I(\lambda)$?

Solutions: As in the real case, as $\lambda \gg 1$ the integral will be dominated by the values of the exponential that are higher. However, letting $f(z) = u(z) + iv(z)$ for real-valued $u, v \in \mathbb{R}$, we now have a contribution in the exponential given by the imaginary part of f:

$$I(\lambda) = \int_\gamma dz\ h(z) e^{\lambda u(z) + i\lambda v(z)} = \int_\gamma dz\ h(z) e^{\lambda u(z)} \left(\cos(v(z)) + i \sin(v(z))\right) \tag{3.14}$$

When $\lambda \gg 1$, the imaginary part will contribute with strong oscillations, and on average will cancel even if the exponential is big. Therefore, the regions contributing the most to the integral will be the ones in which f has a constant phase, i.e. $v(z) = \text{constant}$.

(b) Let z_0 be a critical point $f'(z_0) = 0$. Explain why should we choose γ to pass through z_0 following the steepest-descent directions of the real part Re[f]?

Solution: As we have discussed above, we want to choose a path with constant phase. Indeed, as we have shown in Part I d), the path passing through z_0 through the steepest-descent direction is one such path. Moreover, as we discussed in Part I a), the critical point is a saddle, and the the steepest-descent direction precisely goes through a concave trajectory with z_0 as the maximum, allowing us to apply Laplace's method. See Fig. 2.

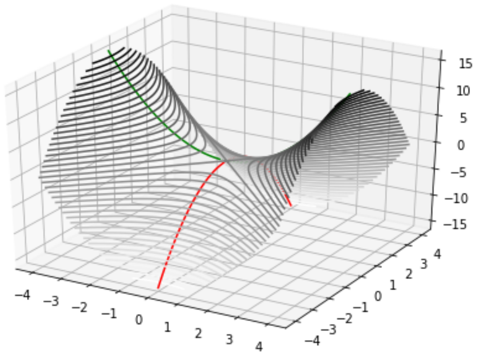


Figure 2: Neighbourhood of a saddle-point, showing in red the direction of steepest-descent and in green the direction of steepest-ascent..

(c) Show that such a γ, we can rewrite the integral as:

$$I(\lambda) = e^{i\lambda \mathrm{Im[f(z_0)]}} \int_\gamma \mathrm{d}z\ h(z) e^{\lambda \mathrm{Re[f(z)]}} \tag{3.15}$$

Solution: Let γ be the curve passing through the critical point z_0 in the steepest-descent direction, and let $f(z) = u(z) + iv(z)$. As we have shown in part I d), $v(z) = \text{constant} = \mathrm{v(z_0)}$ along γ. Therefore:

$$I(\lambda) = e^{i\lambda v(z_0)} \int_\gamma \mathrm{d}z\ h(z) e^{\lambda u(z)}. \tag{3.16}$$

(d) Let $\gamma(t) = x(t) + iy(t)$ for $t \in [a, b]$ be a parametrisation of the curve passing through $z_0 = \gamma(t_0)$ through the steepest-descent direction of $\mathrm{Re[f]}$. Letting $f(t) = f(\gamma(t))$, $h(t) = h(\gamma(t))$, $u(t) = \mathrm{Re[f(t)]}$ and $v(t) = \mathrm{Im[f(t)]}$, show that the problem boils down to the evaluation of the following integral:

$$\int_a^b \mathrm{d}t\ \gamma'(t) h(t) e^{\lambda u(t)} \tag{3.17}$$

Solution: Let $\gamma(t) = x(t) + iy(t)$ for $t \in [a, b]$ be a parametrisation of γ, define $h(t) \equiv h(\gamma(t))$, $u \equiv u(\gamma(t))$, $v(t) \equiv v(\gamma(t))$. Evaluating the expression above at the parametrisation leads to:

$$I(\lambda) = e^{i\lambda v(t_0)} \int_a^b \mathrm{d}t\ \gamma'(t) h(t) e^{\lambda u(t)}. \tag{3.18}$$

Part III: Back to Laplace's method

(a) Suppose $h(t_0) \neq 0$, and note we can choose a parametrisation of γ such that $\gamma'(t_0) \neq 0$. Use Laplace's method to show that $I(\lambda)$ admits the following asymptotic expansion for $\lambda \gg 1$:

$$I(\lambda) \asymp h(t_0)\gamma'(t_0)\sqrt{\frac{2\pi}{-\lambda u''(t_0)}}e^{\lambda f(t_0)} \tag{3.19}$$

Solution: As we have discussed in part II a), in the steepest descent we go through a concave direction in the saddle. Therefore, $z_0 = \gamma(t_0)$ is the only maximum along γ. Applying the result of Exercise 1.2, eq. (2.14) with $c = t_0$ we have:

$$I(\lambda) \underset{\lambda \gg 1}{=} h(t_0)\gamma'(t_0)\sqrt{\frac{2\pi}{-u''(t_0)}}e^{\lambda(u(t_0)+iv(t_0))} = h(t_0)\gamma'(t_0)\sqrt{\frac{2\pi}{-u''(t_0)}}e^{\lambda f(t_0)} \tag{3.20}$$

where we have assumed $h(t_0) \neq 0$.

(b) Write the second derivative of $f(t)$ with respect to t and show that at the critical point z_0 we have:

$$\frac{\mathrm{d}^2 f(t_0)}{\mathrm{d}t^2} = \gamma'(t_0)^2 \frac{\mathrm{d}^2 f(z_0)}{\mathrm{d}z^2} \tag{3.21}$$

Solution: Recall that $f(t) \equiv f(\gamma(t))$. Therefore, by the chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \gamma'(t)\frac{\mathrm{d}f}{\mathrm{d}z}, \qquad \Rightarrow \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \gamma''(t)\frac{\mathrm{d}f}{\mathrm{d}z} + \gamma'(t)^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} \tag{3.22}$$

At the critical point we have $f'(z_0) = 0$, yielding the desired result.

(c) Show that the second derivative $f''(t_0)$ is necessarily real and negative. Conclude that:

$$u''(t_0) = -|f''(z_0)||\gamma'(t_0)|^2 \tag{3.23}$$

Solution: Recall the result from Part I d). Along the steepest-descent solution we had $f''(z_0) = -\rho$ with $\rho = |f''(z_0)| > 0$. Therefore, letting $f(z) = u(z) + i(z)$, inserting this in the above result gives:

$$u''(t_0) = -|f''(z_0)||\gamma'(t_0)|^2 \tag{3.24}$$

(d) Let θ be the angle between the curve γ and the real axis at the critical point z_0, see figure below. Show that:

$$\gamma'(t_0) = |\gamma'(t_0)|e^{i\theta} \tag{3.25}$$

Solution: This follows from the fact that close to the critical point $z_0 = \gamma(t_0)$

$$z - z_0 \equiv \gamma(t) - \gamma(t_0) = \gamma'(t_0)(t - t_0) + O\left((t - t_0)^2\right) \tag{3.26}$$

which implies that the phase of $\gamma'(t_0)$ is the same as the phase of $(z - z_0)$ at a neighbourhood of z_0. From Part I d), the phase of $(z - z_0)$ is precisely chosen to be the steepest-descent direction θ.

(e) Letting $f''(z_0) = |f''(z_0)|e^{i\alpha}$, show that $\theta = \frac{1}{2}(\pi - \alpha)$ or $\theta = \frac{1}{2}(\pi - \alpha) + \pi$ depending on the orientation of the curve γ.

Solution: Recall your result from part I d) showing that the steepest-descent direction is given by $\theta = \frac{1}{2}(\pi - \alpha) + \pi k$ for $k \in \mathbb{Z}$. Choosing $k = 0$ or $k = 1$ fix the direction of the curve.

(f) Conclude that:

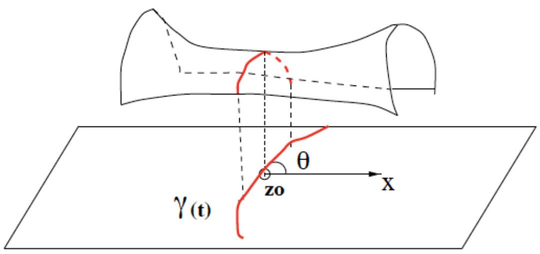
$$I(\lambda) \asymp \pm h(z_0)e^{\lambda f(z_0)}e^{i\frac{\pi-\alpha}{2}}\sqrt{\frac{2\pi}{\lambda|f''(z_0)|}} = h(z_0)e^{\lambda f(z_0)}\sqrt{\frac{2\pi}{-\lambda f''(z_0)}} \tag{3.27}$$

where the $\pm$ is given by the orientation of the steepest-descent curve.

Solution: Using the result of part III c) and the choice $\theta = \frac{1}{2}(\pi - \alpha)$ together with eq. (3.20):

$$I(\lambda) \underset{\lambda\gg1}{=} h(z_0)|\gamma'(t_0)|e^{i\frac{\pi-\alpha}{2}}\sqrt{\frac{2\pi}{\lambda|f''(z_0)|}}e^{\lambda f(z_0)} = h(z_0)\sqrt{\frac{2\pi}{-\lambda f''(z_0)}}e^{\lambda f(z_0)} \tag{3.28}$$

where in the last equality we have used $e^{i\frac{\pi-\alpha}{2}} = \sqrt{-1}e^{-i\alpha/2}$ and have fixed $|\gamma'(t_0)| = 1$. Note that it is always possible to fix the velocity $\gamma'(t)$ of your parametrisation at a point $t_0 \in [a, b]$ to a chosen value v. It just amounts to solving an ordinary differential equation, which given the boundary condition $v \equiv \gamma'(t_0)$ has one and only one solution, see Picard-Lindelöf's theorem.



A Binomial theorem

In this appendix we proof the Binomial theorem:

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \tag{A.1}$$

using an inductive argument. First, note that since:

$$\binom{0}{0} = 1, \qquad \binom{1}{0} = 1, \qquad \binom{1}{1} = 1 \tag{A.2}$$

The binomial theorem trivially holds for $n = 0, 1$. Assume it holds for $n > 1$. We want to show that it also holds for $n+1$. Start by writing:

$$(x+y)^{n+1} = (x+y)(x+y)^n \underset{(a)}{=} (x+y)\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}$$
$$= \sum_{i=0}^{n}\binom{n}{i}x^{i+1}y^{n-i} + \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i+1} \tag{A.3}$$

where in (a) we used the Binomial theorem for n. Now let's rewrite each of these terms:

$$\sum_{i=0}^{n}\binom{n}{i}x^{i+1}y^{n-i} = \sum_{i=0}^{n-1}\binom{n}{i}x^{i+1}y^{n-i} + \binom{n}{n}x^{n+1}y^0 \underset{(a)}{=} \sum_{k=1}^{n}\binom{n}{k-1}x^k y^{n-k+1} + x^{n+1} \tag{A.4}$$
$$\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i+1} = \binom{n}{0}x^0 y^{n+1} + \sum_{i=1}^{n}\binom{n}{i}x^i y^{n-i+1} \underset{(b)}{=} y^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n-k+1} \tag{A.5}$$

where in (a) we let $k = i+1$ and in (b) we simply relabelled $k = i$. Putting together,

$$(x+y)^{n+1} = x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]x^k y^{n-k+1} \tag{A.6}$$

Using Pascal's rule (this can be checked explicitly from the definition):

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \tag{A.7}$$

we have:

$$(x+y)^{n+1} = x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n-k+1}$$
$$\underset{(a)}{=} \binom{n+1}{0}x^{n+1} + \binom{n+1}{n+1}y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n-k+1}$$
$$\underset{(b)}{=} \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{(n+1)-k} \tag{A.8}$$

where in (a) we used that $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$ and in (b) we absorbed the terms back to the sum. This concludes the proof.

B Mode of binomial

Here we show that the mode of the binomial distribution equals to its mean. Note this property is far from generic, and is a special feature symmetric distributions such as the normal and binomial.

Let:

$$\mathbb{P}(X = k) = \binom{n}{k}p^k(1-p)^{n-k} \tag{B.1}$$

be the probability mass function of the binomial random variable X with parameters $n > 0$ and $0 \leq p \leq 1$. The mode of a distribution is defined as:

$$\text{mode} = \underset{0\leq \mathrm{k}\leq \mathrm{n}}{\text{argmin}}\ \mathbb{P}(\mathrm{X} = \mathrm{k}). \tag{B.2}$$

For a discrete random variable, this definition is equivalent to saying that $k_\star$ is the mode if:

$$\mathbb{P}(X = k_\star) \geq \mathbb{P}(X = k_\star + 1), \qquad \mathbb{P}(X = k_\star) \geq \mathbb{P}(X = k_\star - 1) \tag{B.3}$$

Before evaluating the mode, note that the binomial distribution satisfies:

$$\frac{\mathbb{P}(X = k)}{\mathbb{P}(X = k+1)} = \frac{k+1}{n-k}\frac{1-p}{p}, \qquad \frac{\mathbb{P}(X = k)}{\mathbb{P}(X = k-1)} = \frac{n-k+1}{k}\frac{p}{1-p} \tag{B.4}$$

Consider now $k_\star = \mathbb{E}X = np$ satisfies this property. For simplicity, here we consider the case in which np is an integer, think $p = k/n$ for instance (what happens otherwise?). Looking at the first identity:

$$\frac{\mathbb{P}(X = np)}{\mathbb{P}(X = np+1)} = 1 + \frac{1}{np} \geq 1 \tag{B.5}$$

Now looking at the second:

$$\frac{\mathbb{P}(X = np)}{\mathbb{P}(X = np-1)} = 1 + \frac{1}{n(1-p)} \geq 1. \tag{B.6}$$

which is what we wanted to show.