Solutions for Week 11

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1 Exercise 12.1

The p-spin model is one of the cornerstone of spin glass theory. It is defined as follows: there are 2^N possible configurations for the N spins variables $S_i = \pm 1$, and the Hamiltonian is given by all possible p-body (or p-upplet) interactions:

$$\mathcal{H} = -\sum_{i_1 < i_2 < \ldots < i_p} J_{i_1,\ldots,i_p} S_{i_1} \ldots S_{i_p} \tag{1.1}$$

with

$$P(J) = \sqrt{\frac{\pi p!}{N^{p-1}}} e^{-\frac{N^{p-1}}{p!} J^2} \tag{1.2}$$

1. Computing the moment of the partition function using Gaussian integrals, show that

$$\mathbb{E}[Z^n] = \sum_{\{S_i^a\},\{S_i^b\},\ldots,\{S_i^n\}} \exp\left[\frac{\beta^2}{4N^{p-1}} \sum_{a,b} \left(\sum_i S_i^a S_i^b\right)^p\right] \tag{1.3}$$

Solution: The replica computation for the p-spin model is very similar to the one for the the Curie-Weiss model we saw in lectures. Let's recall the main steps. First, define the Gibbs-Boltzmann measure over spin configurations $\boldsymbol{S} \in \{-1,1\}^N$:

$$\mathbb{P}_\beta(\boldsymbol{S} = \boldsymbol{s}) = \frac{1}{\mathcal{Z}_\beta(J)} e^{-\beta \mathcal{H}(\boldsymbol{s})} = \frac{1}{\mathcal{Z}_\beta(J)} e^{\beta \sum\limits_{i_1 < i_2 < \ldots < i_p} J_{i_1,\ldots,i_p} s_{i_1} \ldots s_{i_p}} \tag{1.4}$$

where the normalisation $\mathcal{Z}_\beta$ (a.k.a. the partition sum) is explicitly given by:

$$\mathcal{Z}_\beta(J) = \sum_{\boldsymbol{s} \in \{-1,1\}^N} e^{\beta \sum\limits_{i_1 < i_2 < \ldots < i_p} J_{i_1,\ldots,i_p} s_{i_1} \ldots s_{i_p}} \tag{1.5}$$

The goal of the replica computation is to compute the averaged free energy density in the thermodynamic limit $N \to \infty$:

$$f_\beta = -\lim_{N\to\infty} \frac{1}{N} \mathbb{E}_J \log \mathcal{Z}_\beta(J). \tag{1.6}$$

Note that we expect $\frac{1}{N} \log \mathcal{Z}_\beta$ to concentrate on its average as $N \to \infty$. However, taking the average of a logarithm is hard. Therefore, the second step in the replica computation is to rewrite the free energy density using the replica trick:

$$f_\beta = -\lim_{N\to\infty} \frac{1}{N} \lim_{n\to 0^+} \frac{1}{n} \left[\mathbb{E}_J \mathcal{Z}_\beta^n - 1\right] \tag{1.7}$$

Taking the average over the replicated partition sum is now straightforward:

$$
\begin{aligned}
\mathbb{E}_J \mathcal{Z}_\beta^n &= \mathbb{E}_J \left[\sum_{\boldsymbol{s}\in\{-1,1\}} e^{\beta \sum\limits_{i_1<i_2<\ldots<i_p} s_{i_1}\ldots s_{i_p}} \right] \\
&\overset{(a)}{=} \prod_{a=1}^{n} \sum_{\boldsymbol{s}^a\in\{-1,1\}} \mathbb{E}_J \left[\prod_{i_1,i_2,\ldots,i_p} e^{\frac{\beta}{p!}\sum\limits_{a=1}^{n} J_{i_1,\ldots,i_p} s^a_{i_1}\ldots s^a_{i_p}} \right] \\
&= \prod_{a=1}^{n} \sum_{\boldsymbol{s}^a\in\{-1,1\}} \exp\left[\frac{\beta^2}{4N^{p-1}} \sum_{i_1,i_2,\ldots,i_p} \left(\sum_{a=1}^{n} s^a_{i_1}\ldots s^a_{i_p} \right)^2 \right] \\
&\overset{(b)}{=} \prod_{a=1}^{n} \sum_{\boldsymbol{s}^a\in\{-1,1\}} \exp\left[\frac{\beta^2 N}{4} \sum_{a,b=1}^{n} \left(\frac{\boldsymbol{s}^a\cdot\boldsymbol{s}^b}{N} \right)^p \right]
\end{aligned} \tag{1.8}
$$

where in (a) we used the independence of $J_{i_1\cdots i_p}$ to take the average, using the fact that:

$$
\mathbb{E}_{x\sim\mathcal{N}(0,\Delta)}\left[e^{\lambda x}\right] = e^{\frac{1}{2}\Delta\lambda^2} \tag{1.9}
$$

see discussion in eq. (1.39) for more detail. In (b) we noticed that the indices $i_1,\cdots,i_p$ decouple, which gives the result.

2. introducing delta functions to fix the overlap and taken its Fourier transform, show that

$$
\mathbb{E}[Z^n] \approx \int \prod_{a<b} dq_{a,b} d\hat{q}_{a,b} e^{-NG(Q,\hat{Q})} \tag{1.10}
$$

with

$$
G(Q,\hat{Q}) = -n\frac{\beta^2}{4} - \frac{\beta^2}{2}\sum_{a<b} q^p_{a,b} + i\sum_{a<b}\hat{q}_{a,b}q_{a,b} - \log\left[\sum_{\{S_i^a,\ldots,S_i^n\}} e^{\sum_{a<b} i\hat{q}_{a,b}\sum_i S_i^a S_i^b} \right] \tag{1.11}
$$

Solution: Define the usual overlap between two spin configurations:

$$
q^{ab} = \frac{1}{N}\boldsymbol{s}^a\cdot\boldsymbol{s}^b \tag{1.12}
$$

such that:

$$
\mathbb{E}_J \mathcal{Z}_\beta^n = \prod_{a=1}^{n} \sum_{\boldsymbol{s}^a\in\{-1,1\}} \exp\left[\frac{\beta^2 N}{4} \sum_{a,b=1}^{n} \left(q^{ab}\right)^p \right] \tag{1.13}
$$

We now would like to get rid of the sum over the spin configurations. For that, introduce a delta function to free the overlap parameters and rewrite it in Fourier space:

$$
1 \propto \int_{\mathbb{R}^n} \prod_{1\le a\le b\le n} \mathrm{d}q^{ab} \prod_{1\le a\le b\le n} \delta\left(\boldsymbol{s}^a\cdot\boldsymbol{s}^b - Nq^{ab}\right) = \int_{\mathbb{R}^n} \prod_{1\le a\le b\le n} \frac{\mathrm{d}q^{ab}\mathrm{d}\hat{q}^{ab}}{2\pi} e^{i\sum\limits_{1\le a\le b\le n}\hat{q}^{ab}\left(\boldsymbol{s}^a\cdot\boldsymbol{s}^b - Nq^{ab}\right)} \tag{1.14}
$$

Inserting this in the expression for the replicated partition function allow us to write:

$$
\begin{aligned}
\mathbb{E}_J \mathcal{Z}_\beta^n &\propto \prod_{a=1}^n \sum_{\boldsymbol{s}^a \in \{-1,1\}} \int_{\mathbb{R}^n} \prod_{1\leq a\leq b\leq n} \frac{\mathrm{d}q^{ab}\mathrm{d}\hat{q}^{ab}}{2\pi} e^{i\sum\limits_{1\leq a\leq b\leq n} \hat{q}^{ab}\left(\boldsymbol{s}^a\cdot\boldsymbol{s}^b - Nq^{ab}\right) + \frac{\beta^2 N}{4}\sum\limits_{a,b=1}^n \left(q^{ab}\right)^p} \\
&= \int_{\mathbb{R}^n} \prod_{1\leq a\leq b\leq n} \frac{\mathrm{d}q^{ab}\mathrm{d}\hat{q}^{ab}}{2\pi} e^{-iN\sum\limits_{1\leq a\leq b\leq n} \hat{q}^{ab}q^{ab} + \frac{\beta^2 N}{4}\sum\limits_{a,b=1}^n \left(q^{ab}\right)^p} \prod_{a=1}^n \sum_{\boldsymbol{s}^a \in \{-1,1\}} e^{i\sum\limits_{1\leq a\leq b\leq n} \hat{q}^{ab}\boldsymbol{s}^a\cdot\boldsymbol{s}^b} \qquad (1.15) \\
&= \int_{\mathbb{R}^n} \prod_{1\leq a\leq b\leq n} \frac{\mathrm{d}q^{ab}\mathrm{d}\hat{q}^{ab}}{2\pi} e^{-NG(\{q^{ab},\hat{q}^{ab}\})} \qquad (1.16)
\end{aligned}
$$

where we in the last line we have defined:

$$
G(\{q^{ab},\hat{q}^{ab}\}) = i\sum_{1\leq a\leq b\leq n} \hat{q}^{ab}q^{ab} - \frac{\beta^2}{4}\sum_{a,b=1}^n \left(q^{ab}\right)^p - \frac{1}{N}\log\left\{\prod_{a=1}^n \sum_{\boldsymbol{s}^a\in\{-1,1\}^N} e^{i\sum\limits_{1\leq a\leq b\leq n}\hat{q}^{ab}\boldsymbol{s}^a\cdot\boldsymbol{s}^b}\right\} \qquad (1.17)
$$

Three simplifications lead to the expressions in the exercise. First, we note that the log term factorise in N:

$$
\begin{aligned}
\frac{1}{N}\log\left[\prod_{a=1}^n \sum_{\boldsymbol{s}^a\in\{-1,1\}^N} e^{i\sum\limits_{1\leq a<b\leq n}\sum\limits_{i=1}^N \hat{q}^{ab}s_i^a s_i^b}\right] &= \frac{1}{N}\log\left[\left(\prod_{a=1}^n \sum_{s^a\in\{-1,1\}} e^{i\sum\limits_{1\leq a<b\leq n}\hat{q}^{ab}s^a s^b}\right)^N\right] \\
&= \log\left(\prod_{a=1}^n \sum_{s^a\in\{-1,1\}} e^{i\sum\limits_{1\leq a<b\leq n}\hat{q}^{ab}s^a s^b}\right) \qquad (1.18)
\end{aligned}
$$

Second, we note that since $\boldsymbol{s}\in\{-1,1\}^N$ we have:

$$
q^{aa} = \frac{1}{N}||\boldsymbol{s}^a||_2^2 = 1 \qquad (1.19)
$$

and therefore:

$$
\frac{\beta^2 N}{4}\sum_{a,b=1}^n \left(q^{ab}\right)^p = \frac{\beta^2}{4}n + \frac{\beta^2}{2}\sum_{1\leq a<b\leq n}\left(q^{ab}\right)^p \qquad (1.20)
$$

Third, we note that this also implies that we don't need a constraint to fix q^{aa}, and without loss of generality we can set $\hat{q}^{aa} = 0$. Putting these three observations together yield the expression given by the exercise.

3. Using the replica method, with the replica symmetric solution, show that the free entropy is given by

$$
f_{\mathrm{RS}} = \frac{\beta^2}{4} + \log 2 \qquad (1.21)
$$

as in the REM. It is possible, of course, to break the symmetry and to obtain the 1RSB solution, which turns out to be correct at low temperature (at least for p large enough, and for a range of temperature).

Solution: Note that in the thermodynamic limit $N \to \infty$, we can apply Laplace's method to compute the integral in the replicated partition sum:

$$\mathbb{E}_J \mathcal{Z}_\beta^n \asymp e^{-NG(\{q_\star^{ab}, \hat{q}_\star^{ab}\})} \tag{1.22}$$

where $(q_\star^{ab}, \hat{q}_\star^{ab})$ are solutions of the following extremisation problem:

$$\underset{q^{ab}, \hat{q}^{ab}}{\text{extr}} \; G(\{q^{ab}, \hat{q}^{ab}\}) \tag{1.23}$$

To solve this extremisation problem, we can look to zero-gradient points of G, which yield a set of self-consistent saddle-point equations:

$$i\hat{q}^{ab} = \frac{\beta^2}{4} p \left(q^{ab}\right)^{p-1}, \qquad q^{ab} = \frac{\prod\limits_{a=1}^{n} \sum\limits_{s^a \in \{-1,1\}} s^a s^b e^{\sum\limits_{1 \le a < b \le n} \hat{q}^{ab} s^a s^b}}{\prod\limits_{a=1}^{n} \sum\limits_{s^a \in \{-1,1\}} e^{i \sum\limits_{1 \le a < b \le n} \hat{q}^{ab} s^a s^b}} \equiv i \langle s^a s^b \rangle_n \tag{1.24}$$

which give us another interpretation of q^{ab} as a two-point correlation function of the replicated system. If we could solve these equations, we would be able to insert the solution back in G and take the $n \to 0^+$ to get the free energy density. However, solving for the $n(n-1)$ variables $q^{ab}, \hat{q}^{ab}$ is intractable. Therefore, we restrict ourselves to searching for solutions which are replica symmetric:

$$q^{ab} = q, \qquad \hat{q}^{ab} = -i\hat{q}, \qquad 1 \le a < b \le n \tag{1.25}$$

Which reduces the number of parameters we need to optimise from $n(n-1)$ to 2, and considerable simplify the expression for G:

$$G_{\text{RS}}^{(n)}(q, \hat{q}) = -n\frac{\beta^2}{4} - \frac{\beta^2 n(n-1)}{4} q^p + \frac{n(n-1)}{2} \hat{q} q - \frac{1}{N} \log \left[\prod_{a=1}^{n} \sum_{s^a \in \{-1,1\}} e^{\hat{q} \sum\limits_{1 \le a < b \le n} s^a s^b} \right] \tag{1.26}$$

To simplify further, we can decouple the different replicas with the usual Hubbard-Stratonovich transformation:

$$e^{\hat{q} \sum\limits_{1 \le a < b \le n} s^a s^b} = \mathbb{E}_{z \sim \mathcal{N}(0,1)} \left[e^{\sqrt{\hat{q}} z \sum\limits_{a=1}^{n} s^a - \frac{n}{2} \hat{q}} \right] \tag{1.27}$$

Which finally allow us to factorise in replica space:

$$\begin{aligned}
\log \left(\prod_{a=1}^{n} \sum_{s^a \in \{-1,1\}} e^{\hat{q} \sum\limits_{1 \le a < b \le n} s^a s^b} \right) &= -\frac{n}{2} \hat{q} + \log \left(\prod_{a=1}^{n} \sum_{s^a \in \{-1,1\}} \mathbb{E}_{z \sim \mathcal{N}(0,1)} \left[\prod_{a=1}^{n} e^{\sqrt{\hat{q}} z s^a} \right] \right) &(1.28)\\
&= -\frac{n}{2} \hat{q} + \log \mathbb{E}_{z \sim \mathcal{N}(0,1)} \left(2 \cosh\left(\sqrt{\hat{q}} z \right) \right)^n &(1.29)
\end{aligned}$$

and to take the $n \to 0^+$ limit explicitly:

$$G_{\text{RS}}(q, \hat{q}) \equiv \lim_{n \to 0^+} \frac{1}{n} G_{\text{RS}}^{(n)}(q, \hat{q}) = -\frac{\beta^2}{4} + \frac{\beta^2}{4} q^p - \frac{1}{2} q \hat{q} + \frac{\hat{q}}{2} - \mathbb{E}_{z \sim \mathcal{N}(0,1)} \left[\log 2 \cosh \left(\sqrt{\hat{q}} z \right) \right] \tag{1.30}$$

The replica symmetric free energy density is therefore given by:

$$f_\beta^{\text{RS}} = -\underset{q, \hat{q}}{\textbf{extr}} \; G_{\text{RS}}(q, \hat{q}) \tag{1.31}$$

To find the extrema, we look for zero gradient points of G_{RS}, which lead to the following set of saddle-point equations:

$$\hat{q} = \frac{\beta^2}{2} p q^{p-1}, \qquad q = 1 - \frac{1}{\sqrt{\hat{q}}} \mathbb{E}_{z\sim\mathcal{N}(0,1)} \left[z \tanh\left(\sqrt{\hat{q}} z\right)\right] \tag{1.32}$$

Using Stein's law, the second saddle-point equation can also be writen as:

$$q = \mathbb{E}_{z\sim\mathcal{N}(0,1)} \left[\tanh^2\left(\sqrt{\hat{q}} z\right)\right] \tag{1.33}$$

Putting together, we can solve for $\hat{q}$ write everything in terms of a single parameter q:

$$f_\beta^{\text{RS}} = -\frac{\beta^2}{4}\left[1 - p q_\star^{p-1} + (p-1) q_\star^p\right] - \mathbb{E}_{z\sim\mathcal{N}(0,1)} \left[\log 2 \cosh\left(\sqrt{\frac{p q_\star^{p-1}}{2}} \beta z\right)\right] \tag{1.34}$$

where $q_\star$ is the fixed point of:

$$q = \mathbb{E}_{z\sim\mathcal{N}(0,1)} \left[\tanh^2\left(\sqrt{\frac{p q^{p-1}}{2}} \beta z\right)\right] \tag{1.35}$$

As for the Random Field Ising Model, we don't have a closed form solution for these equations. However, we can solve them numerically, see Fig.1.35. Note that the paramagnetic solution $q = 0$ is always a fixed point, and in particular it is the global minimum of G_{RS} for $\beta \sim 1$. Therefore, for not so low-temperatures, the replica symmetric free energy is given by:

$$f_\beta^{\text{RS}} = \frac{\beta^2}{4} + \log 2 \tag{1.36}$$

Interestingly, this is the same expression we have found for the replica symmetric free energy of the random energy model. However, it is important to stress that this is only the case at high-temperatures: indeed, for low-temperatures $\beta \gg 1$ the paramagnetic fixed point is not the global minimum.

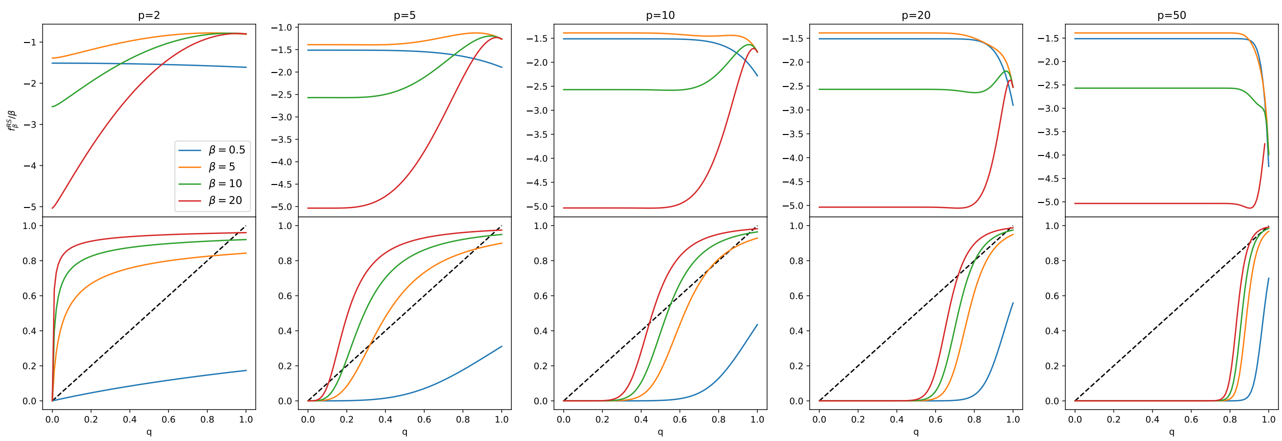

Figure 1: (**Top**) Replica symmetric free energy potential G_{RS} and (**bottom**) right-hand side of fixed-point equation 1.35 as a function of the overlap $q \in [0, 1]$ for different p and inverse temperatures β. In dashed the line $y = q$.

4. It is interesting that the free energy is the same as the one of the REM. Show that, when $p \to \infty$, the energies of the $p - spin$ model becomes uncorrelated and that the $p - spin$ model become the REM in this limit.

Solution: The Random Energy Model (REM) is defined by the distribution of the energies of the 2^N states, which are assumed to be Gaussian random variables with zero mean and variance $N/2$. Therefore, to show that the p-spin model is equivalent to the REM as $p \to \infty$, we need to show that the distribution of the energy of p-spin states coincides with the REM in this limit. Given a state $\boldsymbol{s} \in \{-1, 1\}^N$, its energy under the p-spin model is given by:

$$\mathcal{H}(\boldsymbol{s}) = -\sum_{i_1<i_2<\ldots<i_p} J_{i_1,\ldots,i_p} s_{i_1} \ldots s_{i_p} \tag{1.37}$$

Therefore, the mean of the energy is:

$$\mathbb{E}_J \mathcal{H}(\boldsymbol{s}) = -\sum_{i_1<i_2<\ldots<i_p} \mathbb{E}_J \left[J_{i_1,\ldots,i_p}\right] s_{i_1} \ldots s_{i_p} = 0 \tag{1.38}$$

while the variance (we have computed this already for the replicas) is given by:

$$\begin{aligned}
\mathbb{E}_J \left[\mathcal{H}(\boldsymbol{s}^{(1)})\mathcal{H}(\boldsymbol{s}^{(2)})\right] &= \sum_{i_1<i_2<\ldots<i_p} \sum_{j_1<j_2<\ldots<j_p} \mathbb{E}_J \left[J_{i_1,\ldots,i_p} J_{j_1,\ldots,j_p}\right] s^{(1)}_{i_1} \ldots s^{(1)}_{i_p} s^{(2)}_{j_1} \ldots s^{(2)}_{j_p} \\
&= \frac{p!}{2N^{p-1}} \sum_{i_1<i_2<\ldots<i_p} s^{(1)}_{i_1} \ldots s^{(1)}_{i_p} s^{(2)}_{i_1} \ldots s^{(2)}_{i_p} \\
&\overset{(a)}{=} \frac{1}{2N^{p-1}} \sum_{i_1 \neq i_2 \neq \ldots \neq i_p} s^{(1)}_{i_1} \ldots s^{(1)}_{i_p} s^{(2)}_{i_1} \ldots s^{(2)}_{i_p} \\
&\underset{(b)}{\approx} \frac{1}{2N^{p-1}} \sum_{i_1,i_2,\ldots,i_p} s^{(1)}_{i_1} \ldots s^{(1)}_{i_p} s^{(2)}_{i_1} \ldots s^{(2)}_{i_p} \\
&= \frac{N}{2} \left(\frac{\boldsymbol{s}^{(1)} \cdot \boldsymbol{s}^{(2)}}{N}\right)^p \equiv \frac{N}{2} q(\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)})^p
\end{aligned} \tag{1.39}$$

where in (a) we have used that $p! \sum_{i_1<i_2<\ldots<i_p} = \sum_{i_1 \neq i_2 \neq \ldots \neq i_p}$ because $p!$ gives all the permutation of ordered indices and in (b) that $\sum_{i_1 \neq i_2 \neq \ldots \neq i_p} \approx \sum_{i_1,i_2,\ldots,i_p}$ since the terms with two equal indices are subleading in N. Remember that $q(\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}) \in [0, 1]$, and therefore when $p \to \infty$, we have that $q(\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)})^p = 1$ if $\boldsymbol{s}^{(1)} = \boldsymbol{s}^{(2)}$ and 0 otherwise. Therefore,

$$\lim_{p\to\infty} \mathbb{E}_J \left[\mathcal{H}(\boldsymbol{s}^{(1)})\mathcal{H}(\boldsymbol{s}^{(2)})\right] = \frac{N}{2} \mathbb{I}(\boldsymbol{s}^{(1)} = \boldsymbol{s}^{(2)}) \tag{1.40}$$

which is precisely the variance of the REM. Indeed, it is not hard to show that this implies that their moment generating function also match in this limit, and therefore all the moments coincide.

2 Exercise 12.2

We consider again the REM. Using the replica method, show that at for $\beta \geq \beta_c$, the second moment of the participation ratio is given by

$$\mathbb{E}[Y^2] = \frac{3 - 5m + 2m^2}{3} \tag{2.1}$$

Deduce that Y is not self-averaging. Actually, these results do not depends on the REM, but on the 1RSB structure, and are universal to all 1RSB models.

Solution: Recall the fact that, in the REM, the partition function is defined as:

$$Z = \sum_{i=1}^{2^N} e^{-\beta E_i}, \tag{2.2}$$

where E is a random energy level with distribution:

$$P(E) = \frac{e^{-\frac{E^2}{N}}}{\sqrt{\pi/N}}. \tag{2.3}$$

After computing the 1RSB free-energy in the case of temperatures lower than the critical one, we have introduced the participation ratio Y, defined as:

$$Y = \sum_{i=1}^{2^N} \left(\frac{e^{-\beta E_i}}{Z}\right)^2. \tag{2.4}$$

This quantity can be used to evaluate how many configurations belong to the dominant "state". In fact, if we reorganize the terms in the definition we get:

$$Y = \frac{1}{Z}\sum_{i=1}^{2^N} e^{-\beta E_i}\left(\frac{e^{-\beta E_i}}{Z}\right) = \frac{\sum_{i=1}^{2^N} e^{-\beta E_i}\left(\frac{e^{-\beta E_i}}{Z}\right)}{\sum_{i=1}^{2^N} e^{-\beta E_i}} \tag{2.5}$$

which is the expectation (over the measure of our problem) of the probability of finding a configuration with energy E_i. We also evaluated the first moment of the participation (in the notes) to be equal to:

$$\mathbb{E}\left[Y\right] = 1 - m = 1 - \frac{\beta_c}{\beta} \tag{2.6}$$

below the critical temperature $\beta_c = 2\sqrt{\log 2}$.

Now, we evaluate the second moment with the same procedure. We start from the square of Y:

$$Y^2 = \left(\sum_{i=1}^{2^N}\left(\frac{e^{-\beta E_i}}{Z}\right)^2\right)^2 = Z^{-4}\sum_{j_1,j_2=1}^{2^N} e^{-2\beta(E_{j_1}+E_{j_2})}. \tag{2.7}$$

We need to take the expectation of this quantity w.r.t. the random distribution of the energy levels. Thus, we start by replicating the expression:

$$\mathbb{E}\left[Y^2\right] = \lim_{n\to 0}\mathbb{E}\, Z^{n-4}\sum_{j_1,j_2=1}^{2^N} e^{-2\beta(E_{j_1}+E_{j_2})} = \lim_{n\to 0}\mathbb{E}\sum_{i_1,\dots,i_{n-4}=1}^{2^N} e^{-\beta(E_{i_1}+\dots+E_{i_{n-4}})}\sum_{j_1,j_2=1}^{2^N} e^{-2\beta(E_{j_1}+E_{j_2})}. \tag{2.8}$$

Now we can put the additional terms, summed over j_1 and j_2, in the first multiple sum. However, we also have to add a constraint enforcing that the last two couples of replicas have the same energy:

$$\mathbb{E}\left[Y^2\right] = \lim_{n\to 0}\mathbb{E}\sum_{i_1,\dots,i_n=1}^{2^N} e^{\beta(E_{i_1}+\dots+E_{i_n})}\,\mathbb{1}(i_{n-3}=i_{n-2})\mathbb{1}(i_{n-1}=i_n). \tag{2.9}$$

Now, recall the fact that we defined the overlap parameter for this model precisely as $Q_{a,b} = \mathbb{1}(i_a = i_b)$. We can use the symmetry between the replicas to rewrite the above expression as:

$$\mathbb{E}\left[Y^2\right] = \lim_{n\to 0}\frac{2}{n(n-1)(n-2)(n-3)}\sum_{a\neq b\neq c\neq d}\left\langle Q_{a,b}\,Q_{c,d}\right\rangle. \tag{2.10}$$

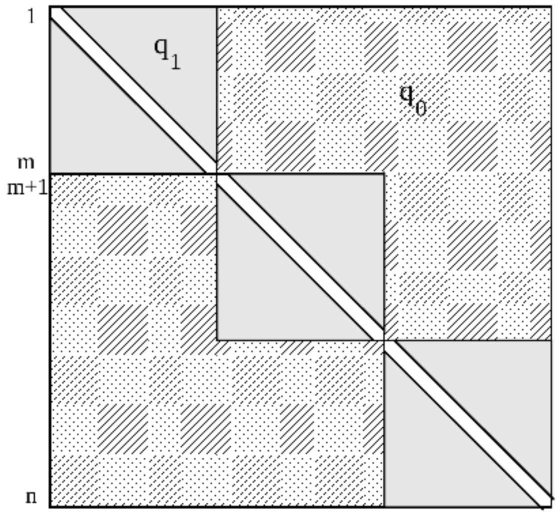

Figure 2: 1RSB overlap matrix, with a block structure with parameters q_1 (diagonal blocks) and q_0 (off-diagonal blocks). In the REM, $q_1 = 1$ and q_0.

The saddle-point approximation we are assuming to hold tells us that the sum is dominated by terms with a specific value for the overlap matrix Q. Below the critical temperature, we have seen that the matrix displays a 1RSB block structure, with n/m square blocks with entry 1 stacked on the diagonal, and zeros elsewhere (figure 2 shows and example).

Thus, the only remaining step is to counting how many distinct products $Q_{a,b}\,Q_{c,d}$ yield a non-zero contribution to the sum we have just written. We get:

$$\mathbb{E}\left[Y^2\right] = \lim_{n\to 0} \frac{\frac{n}{m}m(m-1)\left((m^2 - m - (m-1) - (m-2)) + (\frac{n}{m} - 1)m(m-1)\right)}{n(n-1)(n-2)(n-3)/2}, \tag{2.11}$$

where in the second term we are first picking an entry of $Q_{c,d}$ from the same block as in $Q_{a,b}$, and then from different diagonal blocks. This expression can be simplified to yield the sought result:

$$\mathbb{E}\left[Y^2\right] = \frac{3 - 5m + 2m^2}{3}. \tag{2.12}$$

This value is clearly different from $\mathbb{E}\left[Y\right]^2$, so the variance of the participation ratio is non-zero, thus this quantity does not concentrate in the high limit.