Solutions for Week 7

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1 Exercise 7.1

Consider the rank-one factorization model with vectors X where each component is sampled uniformly from $\mathcal{N}(0,1)$.

a) Using the replica expression for the free entropy, show that the overlap m between the posterior estimate $\langle x \rangle$ and the real value x^* obeys a self consistent equation.

Solution: Recall that we have shown in the lectures that the overlap $m^\star$ between the posterior estimate and the real value $\boldsymbol{x}^\star \sim \prod_{i=1}^{N} P_x(x_i)$ solves the following extremisation problem:

$$m^\star = \underset{m\in[0,1]}{\text{argmax}} \underbrace{\left\{ -\frac{\lambda}{4}m^2 + \mathbb{E}_{x^\star,z} \log \int \mathrm{d}x \; P_x(x) e^{-\frac{\lambda m}{2}x^2 + (\lambda m x^\star + \sqrt{\lambda m} z)x} \right\}}_{\equiv \Phi_{\text{RS}}(m)} \tag{1.1}$$

where $x^\star \sim P_x$ and $z \sim \mathcal{N}(0,1)$ is an effective noise variable. To find the set of maxima of the replica symmetric potential Φ_{RS}, we look at its zero derivative points. This give us the following self-consistent equation for m:

$$\frac{\partial \Phi_{\text{RS}}}{\partial m} = 0 \qquad \Leftrightarrow \qquad m = \mathbb{E}_{x_*,z} \frac{\int \mathrm{d}x \; P_x(x) \; x x^* e^{\frac{-\lambda m}{2}x^2 + (\lambda m x^* - \sqrt{\lambda m} z)x}}{\int \mathrm{d}x \; P_x(x) \; e^{\frac{-\lambda m}{2}x^2 + (\lambda m x^* - \sqrt{\lambda m} z)x}} \tag{1.2}$$

In particular, for $P_x = \mathcal{N}(0,1)$ the integrals can be explicitly done:

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{d}x \; \mathcal{N}(x|0,1) e^{\frac{-\lambda m}{2}x^2 + (\lambda m x^* - \sqrt{\lambda m} z)x} &= \frac{1}{\sqrt{1+\lambda m}} e^{\frac{(m x^\star + z\sqrt{\lambda m})^2}{2(1+\lambda m)}} \\ \int_{\mathbb{R}} \mathrm{d}x \; x x^\star \mathcal{N}(x|0,1) e^{\frac{-\lambda m}{2}x^2 + (\lambda m x^* - \sqrt{\lambda m} z)x} &= \frac{1}{\sqrt{1+\lambda m}} e^{\frac{(m x^\star + z\sqrt{\lambda m})^2}{2(1+\lambda m)}} \frac{x^\star(\lambda m x^\star + \sqrt{\lambda m} z)}{1+\lambda m} \end{aligned} \tag{1.3}$$

Therefore, taking the ratio and the average over $x^\star$ and z, we obtain the following self-consistent equation:

$$m = \frac{\lambda m}{1+\lambda m} \tag{1.4}$$

b) Solve this equation numerically, and show that m is non zero only for SNR $\lambda > 1$.

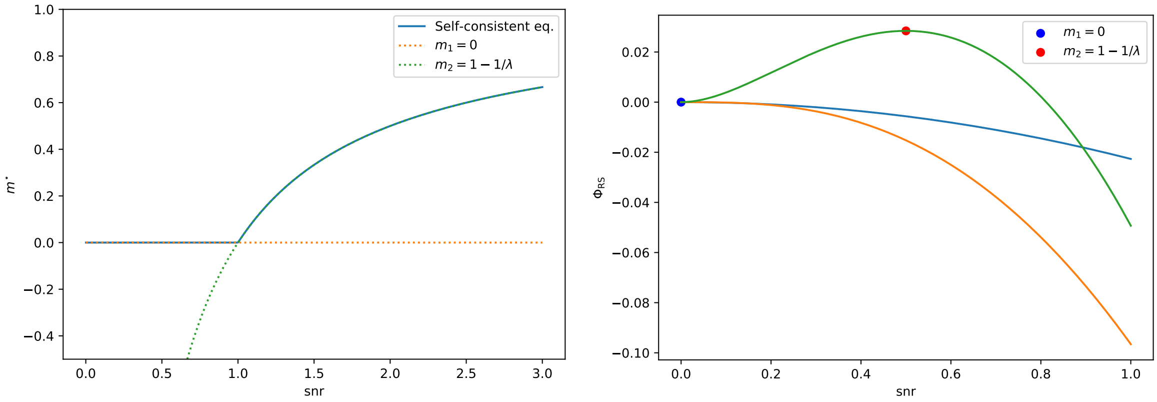

Figure 1: **(Left)** $m^\star$ as a function of the snr λ for spiked matrix estimation with Gaussian spike. **(Right)** Replica symmetric potential as a function of m for different snr λ.

Solution: The self-consistent equation is so simple that it can actually be solved analytically. It has two solutions:

$$m_1 = 0, \qquad\qquad m_2 = 1 - \frac{1}{\lambda} \tag{1.5}$$

Note that $m^\star$, the solution of eq. (1.1), corresponds to the global maximum of the replica potential Φ_{RS}, see Fig. 1. By inserting the two solutions back in Φ_{RS}:

$$\Phi_{\mathrm{RS}}(m_1) = 0, \qquad\qquad \Phi_{\mathrm{RS}}(m_2) = \frac{\lambda(\lambda - 2\log\lambda) - 1}{4\lambda} \tag{1.6}$$

Note that $\Phi_{\mathrm{RS}}(m_2) < 0$ for $\lambda < 1$. Therefore, the global maximum is given by:

$$m^\star = \begin{cases} 0 & \text{for } \lambda \leq 1 \\ 1 - \frac{1}{\lambda} & \text{for } \lambda > 1 \end{cases} \tag{1.7}$$

This corresponds to the following MMSE:

$$\mathrm{mmse}(\lambda) \equiv 1 - \mathrm{m}^\star = \begin{cases} 0 & \text{for } \lambda \leq 1 \\ \frac{1}{\lambda} & \text{for } \lambda > 1 \end{cases} \tag{1.8}$$

This result is quite intuitive: the error decays as λ^{-1} as we increase the signal strength λ. This is exactly what we find by solving the self-consistent equations numerically (see notebook).

c) Once this is done, perform simulations of the model by creating matrices

$$\mathbf{Y} = \sqrt{\frac{\lambda}{N}} \underbrace{\mathbf{x}^*\mathbf{x}^{*\intercal}}_{N\times N \text{ rank-one matrix}} + \underbrace{\xi}_{\text{symmetric iid noise}}$$

and compare the MMSE obtained with this approach with the one of any algorithm you may invent to solve the problem. A classical algorithm for instance, is to use as an estimator the eigenvector of Y corresponding to its largest eigenvalue.

Solution: See notebook.

2 Exercise 7.2

Consider the rank-one factorization model with vectors X where each components is

model 1: sampled uniformly from ± 1

model 2: sampled uniformly from ± 1 (with probability ρ), otherwise 0 (with probability $1-\rho$)

a) Using the replica expression for the free entropy, show that the overlap m between the posterior means estimate $\langle x \rangle$ and the real value x^* obeys a self consistent equation.

Solution: Let's start by looking at model 1. As we have shown in Exercise 7.1, the maxima m^* of the replica symmetric potential satisfy a self-consistent equation which we derived in full generality for any signal distribution P_x, see eq. (1.2). In particular, for model 1 we have $P_x(x) = \frac{1}{2}(\delta_{x,1} + \delta_{x,-1})$. This give us:

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x\,(\delta_{x,1}+\delta_{x,-1})\,e^{\frac{-\lambda m}{2}x^2+(\lambda m x^* - \sqrt{\lambda m}z)x} &= \frac{1}{2}e^{-\frac{\lambda m}{2}}\left(e^{\left(\lambda m x^\star - \sqrt{\lambda m}z\right)} + e^{-\left(\lambda m x^\star - \sqrt{\lambda m}z\right)}\right) \\
&= e^{-\frac{\lambda m}{2}}\cosh\left(\lambda m x^\star - \sqrt{\lambda m}z\right) \\
\frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x\; x x^\star\,(\delta_{x,1}+\delta_{x,-1})\,e^{\frac{-\lambda m}{2}x^2+(\lambda m x^* - \sqrt{\lambda m}z)x} &= \frac{x^\star}{2}e^{-\frac{\lambda m}{2}}\left(e^{\left(\lambda m x^\star - \sqrt{\lambda m}z\right)} - e^{-\left(\lambda m x^\star - \sqrt{\lambda m}z\right)}\right) \\
&= e^{-\frac{\lambda m}{2}}x^\star\sinh\left(\lambda m x^\star - \sqrt{\lambda m}z\right) \qquad (2.1)
\end{aligned}$$

Putting together, the self-consistent equation for model 1 is simply given by:

$$m = \mathbb{E}_{z\sim\mathcal{N}(0,1)}\tanh\left(\lambda m + \sqrt{\lambda m}z\right) \qquad (2.2)$$

Similarly, for model 2 we have $P_x(x) = \frac{\rho}{2}(\delta_{x,1}+\delta_{x,-1}) + (1-\rho)\delta_{x,0}$. This will add an extra term to the integrals of eq. (1.2):

$$\int_{\mathbb{R}} \mathrm{d}x\; P_x(x)e^{\frac{-\lambda m}{2}x^2+(\lambda m x^* - \sqrt{\lambda m}z)x} = \rho e^{-\frac{\lambda m}{2}}\cosh\left(\lambda m x^\star - \sqrt{\lambda m}z\right) + (1-\rho) \qquad (2.3)$$

$$\int_{\mathbb{R}} \mathrm{d}x\; P_x(x)x x^\star e^{\frac{-\lambda m}{2}x^2+(\lambda m x^* - \sqrt{\lambda m}z)x} = \rho e^{-\frac{\lambda m}{2}}x^\star\sinh\left(\lambda m x^\star - \sqrt{\lambda m}z\right) \qquad (2.4)$$

Putting together, the self-consistent equation for m reads:

$$\begin{aligned}
m &= \mathbb{E}_{z,x^\star}\frac{\rho e^{-\frac{\lambda m}{2}}x^\star\sinh\left(\lambda m x^\star - \sqrt{\lambda m}z\right)}{\rho e^{-\frac{\lambda m}{2}}\cosh\left(\lambda m x^\star - \sqrt{\lambda m}z\right) + (1-\rho)} \\
&= \mathbb{E}_{z\sim\mathcal{N}(0,1)}\left[\rho^2\frac{e^{-\frac{\lambda m}{2}}\sinh\left(\lambda m + \sqrt{\lambda m}z\right)}{(1-\rho) + \rho e^{-\frac{\lambda m}{2}}\cosh\left(\lambda m + \sqrt{\lambda m}z\right)}\right] \qquad (2.5)
\end{aligned}$$

b) Solve this equation numerically, and show that m is non zero only for SNR $\lambda > 1$, for model 1 and for a non-trivial critical value for model 2. Check also that, for ρ small enough, the transition is a first order one for model 2.

Solution: See notebook.